Classical mechanics on $GL(n, \mathbb{R})$ group and Euler-Calogero-Sutherland model

A.M. Khvedelidze* and D.M. Mladenov†

Bogoliubov Laboratory of Theoretical Physics,

Joint Institute for Nuclear Research, 141980 Dubna, Russia

Relations between the free motion on the $GL^+(n, \mathbb{R})$ group manifold and the dynamics of an n-particle system with spin degrees of freedom on a line interacting with a pairwise $1/\sinh^2 x$ "potential" (Euler-Calogero-Sutherland model) is discussed in the framework of Hamiltonian reduction. Two kinds of reductions of the degrees of freedom are considered: due to the continuous invariance and due to the discrete symmetry. It is shown that after projection on the corresponding invariant manifolds the resulting Hamiltonian system represents the Euler-Calogero-Sutherland model in both cases.

1. INTRODUCTION

In this contribution, we deal with two finite dimensional Hamiltonian systems. The first one is a generalization of the Calogero-Sutherland-Moser [1] model by introducing the internal degrees of freedom [2, 3] described by the Hamiltonian

$$H_{ECS} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{8}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{\sinh^2(x_i - x_j)} \tag{1}$$

with canonical pairs (x_i, p_i) obeying the nonvanishing Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}\,, \tag{2}$$

and "internal" variables l_{ab} satisfy the $SO(n, \mathbb{R})$ Poisson bracket algebra

$$\{l_{ab}, l_{cd}\} = \delta_{bc} l_{ad} + \delta_{ad} l_{bc} - \delta_{ac} l_{bd} - \delta_{bd} l_{ac}\,. \tag{3}$$

*Department of Theoretical Physics, A. Razmadze Mathematical Institute, GE-380093 Tbilisi, Georgia.

†Electronic address: mladim@thsun1.jinr.ru

The dynamics of the second system is given in terms of geodesic motion on the $GL(n,\mathbb{R})$ group manifold. The corresponding Lagrangian based on the bi-invariant metric on $GL(n,\mathbb{R})$ is given by [4, 5]

$$L_{GL} = \frac{1}{2}\,\mathrm{tr}\left(\dot{g}g^{-1}\right)^2 \,, \tag{4}$$

where $g \in GL(n,\mathbb{R})$, and the dot over the symbols means differentiation with respect to time. Below we shall represent the Hamiltonian system corresponding to this Lagrangian (4) in terms of a special parameterization adapted to the action of the symmetry group of the system. We demonstrate that the resulting Hamiltonian is a generalization of the Euler-Calogero-Sutherland model (1) with two types of internal degrees of freedom. Performing the Hamiltonian reduction owing to two types of symmetry: continuous and discrete, we show how to arrive at the conventional Hamiltonian of the Euler-Calogero-Sutherland model (1).

2. BI-INVARIANT GEODESIC MOTION ON THE GROUP MANIFOLD

2.1.. Explicit integration of the classical equation of motion

The Euler-Lagrange equation following from the Lagrangian (4) can be represented as

$$\frac{d}{dt}\left(g^{-1}\dot{g}\right) = 0\,. \tag{5}$$

This form demonstrates their explicit integrability

$$g(t) = g(0)\exp\left(tJ\right) \tag{6}$$

with two arbitrary constant matrices $g(0)$ and J.

2.2.. Hamiltonian in terms of special coordinates

The canonical Hamiltonian corresponding to the bi-invariant Lagrangian (4) reads

$$H_{GL} = \frac{1}{2}\,\mathrm{tr}\left(\pi^T g\right)^2 \,. \tag{7}$$

The nonvanishing Poisson brackets between the fundamental phase space variables are

$$\{g_{ab}\,, \pi_{cd}\} = \delta_{ab}\,\delta_{cd}\,. \tag{8}$$

To find out the relation to the conventional Euler-Calogero-Sutherland model (1), it is convenient to use the polar decomposition [6] for an arbitrary element of $GL(n,\mathbb{R})$. For the sake of technical simplicity we investigate in details the group $GL(3,\mathbb{R})$ hereinafter, i.e.

$$g = OS\,, \tag{9}$$

where S is a positive definite 3×3 symmetric matrix, and $O(\phi_1,\phi_2,\phi_3) = e^{\phi_1 J_3}e^{\phi_2 J_1}e^{\phi_3 J_3}$ is an orthogonal matrix with $SO(3,\mathbb{R})$ generators $(J_a)_{ik} = \varepsilon_{iak}$. Since the matrix g represents an element of $GL(n,\mathbb{R})$ group, we can treat the polar decomposition (9) as a uniquely invertible transformation from the configuration variables g to a new set of six Lagrangian coordinates S_{ij} and three coordinates ϕ_i. The induced transformation of momenta to new canonical pairs (S_{ab}, P_{ab}) and (ϕ_a, P_a) is

$$\pi = O\left(P - k_a J_a\right)\,, \tag{10}$$

where

$$k_a = \gamma^{-1}_{ab}\left(\eta^L_b - \varepsilon_{bmn}\left(SP\right)_{mn}\right)\,. \tag{11}$$

Here η^L_a are three left-invariant vector fields on $SO(3,\mathbb{R})$

$$\begin{aligned}
\eta^L_1 &= -\frac{\sin\phi_3}{\sin\phi_2}\,P_1 - \cos\phi_3\,P_2 + \cot\phi_2\sin\phi_3\;P_3\,,\\
\eta^L_2 &= -\frac{\cos\phi_3}{\sin\phi_2}\,P_1 + \sin\phi_3\,P_2 + \cot\phi_2\cos\phi_3\;P_3\,,\\
\eta^L_3 &= -P_3
\end{aligned} \tag{12}$$

and $\gamma_{ik} = S_{ik} - \delta_{ik}\ \mathrm{tr}S$. In terms of the new variables, the canonical Hamiltonian takes the form

$$H_{GL} = \frac{1}{2}\mathrm{tr}\left(PS\right)^2 + \frac{1}{2}\mathrm{tr}\left(J_aSJ_bS\right)k_ak_b\,. \tag{13}$$

2.2.1.. Restriction of the Hamiltonian to the Principal orbit

The system (13) is invariant under the orthogonal transformations $S' = R^T\,S\,R$, and the orbit space is given as a quotient space $\mathcal{S}/SO(3,\mathbb{R})$. The quotient space $\mathcal{S}/SO(3,\mathbb{R})$ is a stratified manifold; orbits with the same isotropy group are collected into *strata* and uniquely parameterized by the set of ordered eigenvalues of the matrix S $x_1 \leq x_2 \leq x_3$. The strata are classified according to the isotropy groups which are determined by the degeneracies of the matrix eigenvalues:

1. *Principal orbit-type stratum*, when all eigenvalues are unequal $x_1 < x_2 < x_3$, with the smallest isotropy group $Z_2 \otimes Z_2$.

2. *Singular orbit-type strata* forming the boundaries of orbit space with

   (a) two coinciding eigenvalues (e.g. $x_1 = x_2$), when the isotropy group is $SO(2) \otimes Z_2$.

   (b) all three eigenvalues are equal ($x_1 = x_2 = x_3$), here the isotropy group coincides with the isometry group $SO(3, \mathbb{R})$.

Now we shall at first restrict ourselves to the investigation of dynamics which takes place on the *principal* orbits. To write down the Hamiltonian describing the motion on the principal orbit stratum, we introduce the coordinates along the slices x_i and along the orbits χ. Namely, since the matrix S is positive definite and symmetric, we use the main-axes decomposition in the form

$$S = R^T(\chi) e^{2X} R(\chi) , \tag{14}$$

where $R(\chi) \in SO(3, \mathbb{R})$ is an orthogonal matrix parameterized by three Euler angles $\chi = (\chi_1, \chi_2, \chi_3)$, and the matrix e^{2X} is a diagonal $e^{2X} = \operatorname{diag} \| e^{2x_1}, e^{2x_2}, e^{2x_3} \|$. The original physical momenta P_{ik} are expressed in terms of the new canonical pairs (x_i, p_i) and (χ_i, p_{χ_i}) as

$$P = R^T e^{-X} \left(\sum_{a=1}^{3} \bar{\mathcal{P}}_a \bar{\alpha}_a + \sum_{a=1}^{3} \mathcal{P}_a \alpha_a \right) e^{-X} R , \tag{15}$$

with

$$\bar{\mathcal{P}}_a = \frac{1}{2} p_a , \tag{16}$$

$$\mathcal{P}_a = -\frac{1}{4} \frac{\xi_a^R}{\sinh(x_b - x_c)} , (\text{cyclic} \quad \text{permutation} \ \ a \neq b \neq c) . \tag{17}$$

In the representation (15), we introduce the orthogonal basis for the symmetric 3×3 matrices $\alpha_A = (\overline{\alpha}_i, \ \alpha_i)$ $i = 1, 2, 3$ with the scalar product

$$\operatorname{tr}(\bar{\alpha}_a \, \bar{\alpha}_b) = \delta_{ab} , \quad \operatorname{tr}(\alpha_a \, \alpha_b) = 2\delta_{ab} , \quad \operatorname{tr}(\bar{\alpha}_a \, \alpha_b) = 0$$

and the $SO(3, \mathbb{R})$ right-invariant Killing vectors

$$\xi_1^R = -p_{\chi_1} , \tag{18}$$

$$\xi_2^R = \sin \chi_1 \cot \chi_2 \ p_{\chi_1} - \cos \chi_1 \ p_{\chi_2} - \frac{\sin \chi_1}{\sin \chi_2} \ p_{\chi_3} , \tag{19}$$

$$\xi_3^R = - \ \cos \chi_1 \cot \chi_2 \ p_{\chi_1} - \sin \chi_1 \ p_{\chi_2} + \frac{\cos \chi_1}{\sin \chi_2} \ p_{\chi_3} . \tag{20}$$

After passing to these main-axes variables, the canonical Hamiltonian reads

$$H_{GL} = \frac{1}{8}\sum_{a=1}^{3} p_a^2 + \frac{1}{16}\sum_{(abc)} \frac{(\xi_a^R)^2}{\sinh^2(x_b - x_c)} - \frac{1}{4}\sum_{(abc)} \frac{\left(R_{ab}\eta_b^L + \frac{1}{2}\xi_a^R\right)^2}{\cosh^2(x_b - x_c)}\,. \tag{21}$$

Here (abc) means cyclic permutations $a \neq b \neq c$. Thus, the integrable dynamical system describing the free motion on principal orbits represents, in the adapted basis, the Generalized Euler-Calogero-Sutherland model. The generalization consists in the introduction of two types of internal dynamical variables ξ and η — "spin" and "isospin" degrees of freedom, interacting with each other. Below, the relations to the standard Euler-Calogero-Sutherland model (1) are demonstrated.

2.2.2.. Restriction of the Hamiltonian to the Singular orbit

The motion on the Singular orbit is modified due to the presence of a continuous isotropy group. In the case of $GL(3,\mathbb{R})$, it is $SO(2) \otimes Z_2$. Applying the same machinery as for the Principal orbits to the two-dimensional orbit $(x_1 = x_2 = x\ , x_3 = y)$, one can derive the Hamiltonian

$$H_{GL}^{(2)} = \frac{1}{8}(p_x^2 + p_y^2) + \frac{g^2}{\sinh^2(x-y)} - \frac{\overline{g}^2}{\cosh^2(x-y)} \tag{22}$$

with two arbitrary constants g^2 and $\overline{g}^2$ related to the value of the spin ξ and isospin η. Due to the translation invariance, the equations of motion are equivalent to the corresponding equations for a one-dimensional problem and thus the system (22) is integrable.

3. REDUCTION TO EULER-CALOGERO-SUTHERLAND MODEL

3.1.. Reduction using discrete symmetries

Now we shall demonstrate how the IIA_3 Euler-Calogero-Sutherland model arises from the canonical Hamiltonian (7) after projection onto a certain invariant submanifold determined by discrete symmetries. Let us impose the condition of symmetry of the matrices $g \in GL(n,\mathbb{R})$

$$\chi_a^{(1)} = \varepsilon_{abc} g_{bc} = 0\,. \tag{23}$$

In order to find an invariant submanifold, it is necessary to supplement the constraints (23) with the new ones

$$\chi_a^{(2)} = \varepsilon_{abc}\pi_{bc} = 0\,. \tag{24}$$

One can check that the surface defined by both constraints (23) and (24) represents an invariant submanifold in the $GL(3,\mathbb{R})$ phase space, and the dynamics of the corresponding induced system is governed by the reduced Hamiltonian

$$H_{GL}\big|_{\chi_a^{(1)}=0,\ \chi_a^{(2)}=0} = \frac{1}{2}\mathrm{tr}\,(PS)^2\ . \tag{25}$$

The matrices S and P are now symmetric nondegenerate matrices, and one can be convinced that this expression leads to the Hamiltonian of the IIA_3 Euler-Calogero- Sutherland model. To verify this statement, it is necessary to note that after projection on the invariant submanifold, the canonical Poisson structure is changed. We have to deal with the new Dirac brackets

$$\{F, G\}_D = \{F, G\}_{PB} - \{F, \chi_a\}C^{-1}_{ab}\{\chi_b, G\} \tag{26}$$

for arbitrary functions on the phase space. In our case, because $C_{ab} = \|\{\chi_a^{(1)}, \chi_b^{(2)}\}\| = 2\delta_{ab}\,,$ the fundamamental Dirac brackets between the main-axes variables are

$$\{x_a, p_b\}_D = \frac{1}{2}\delta_{ab}\,, \quad \{\chi_a, p_{\chi_b}\}_D = \frac{1}{2}\delta_{ab}$$

and the Dirac bracket algebra for the right-invariant vector fields on $SO(3,\mathbb{R})$ reduces to

$$\{\xi^R_a, \xi^R_b\}_D = \frac{1}{2}\varepsilon_{abc}\xi^R_c\,.$$

Thus, after rescaling of the canonical variables, one can be convinced that the reduction via discrete symmetry indeed leads to the IIA_3 Euler-Calogero- Sutherland model.

3.2.. Reduction due to the continuous symmetry

The integrals of motion corresponding to the geodesic motion with respect to the bi-invariant metric on $GL(n,\mathbb{R})$ group are

$$J_{ab} = (\pi^T g)_{ab}. \tag{27}$$

The algebra of this integrals realizes on the symplectic level the $GL(n,\mathbb{R})$ algebra

$$\{J_{ab}, J_{cd}\} = \delta_{bc} J_{ad} - \delta_{ad} J_{cb}\,. \tag{28}$$

After transformation to the scalar and rotational variables, the expressions for J reads

$$J = \sum_{a=1}^{3} R^T \left(p_a \bar{\alpha}_a - i_a \alpha_a - j_a J_a\right) R\,, \tag{29}$$

where

$$i_a = \frac{1}{2}\xi^R_a \coth(x_b - x_c) + \left(R_{ab}\eta^L_b + \frac{1}{2}\xi^R_a\right)\tanh(x_b - x_c) \tag{30}$$

and

$$j_a = R_{ab}\eta^L_b + \xi^R_a\,. \tag{31}$$

When these integrals are used, there appear several ways to choose an invariant manifold and to derive the corresponding reduced system. Let us consider the surface on phase space defined by the constraints

$$\eta^R_a = 0\,. \tag{32}$$

These constraints, in the Dirac's terminology [7, 8] are first class constraints $\{\eta^R_a, \eta^R_b\} = -\epsilon_{abc}\eta^R_c$, and the surface (32) is invariant under the evolution governed by the Hamiltonian

$$\{\eta^R_a, H_{GL}\} = 0\,.$$

Using the relation between left and right-invariant Killing fields $\eta^R_a = O_{ab}\eta^L_b$, we find out that after projection to the constraint surface (32), the Hamiltonian reduces to

$$H_{GL}(\eta^R_a = 0) = \frac{1}{8}\sum_{a}^{3} p_a^2 + \frac{1}{4}\sum_{(abc)} \frac{(\xi^R_a)^2}{\sinh^2 2(x_b - x_c)}\,. \tag{33}$$

After rescaling of the variables $2x_a \to x_a$, one is convinced that the derived Hamiltonian coincides with the Euler-Calogero-Sutherland Hamiltonian (1), where the intrinsic spin variables are $l_{ij} = \epsilon_{ijk}\xi^R_k$. Note that performing the reduction to the surface defined by the vanishing integrals $j_a = 0$, we again arrive at the same Euler-Calogero-Sutherland system.

3.3.. Lax-pair for Generalized Euler-Calogero-Sutherland model

The expressions (29) for the integrals of motion allow us to rewrite the classical equation of motion for the Generalized Euler-Calogero-Sutherland model in the Lax form

$$\dot{L} = [A, L]\,, \tag{34}$$

where 3×3 matrices are given explicitly as

$$L = \begin{pmatrix} p_1 & L_3^+, & L_2^- \\ L_3^-, & p_2 & L_1^+ \\ L_2^+, & L_1^-, & p_3 \end{pmatrix}$$

and

$$A = \tfrac{1}{4}\, e^{+X} \begin{pmatrix} p_1 & -A_3, & A_2 \\ A_3, & p_2, & -A_1 \\ -A_2, & A_1, & p_3 \end{pmatrix} e^{-X}\,,$$

where

$$L_1^{\pm} = -\frac{1}{2}\frac{\xi_1^R}{\sinh(x_2 - x_3)} \pm \frac{R_{1m}\eta_m^L + \frac{1}{2}\xi_1^R}{\cosh(x_2 - x_3)}\,, \tag{35}$$

$$L_2^{\pm} = -\frac{1}{2}\frac{\xi_2^R}{\sinh(x_3 - x_1)} \pm \frac{R_{2m}\eta_m^L + \frac{1}{2}\xi_2^R}{\cosh(x_3 - x_1)}\,, \tag{36}$$

$$L_3^{\pm} = -\frac{1}{2}\frac{\xi_3^R}{\sinh(x_1 - x_2)} \pm \frac{R_{3m}\eta_m^L + \frac{1}{2}\xi_3^R}{\cosh(x_1 - x_2)} \tag{37}$$

and

$$A_1 = \frac{1}{2}\frac{\xi_1^R}{\sinh^2(x_2 - x_3)} - \frac{R_{1m}\eta_m^L + \frac{1}{2}\xi_1^R}{\cosh^2(x_2 - x_3)}\,, \tag{38}$$

$$A_2 = \frac{1}{2}\,\frac{\xi_2^R}{\sinh^2(x_3 - x_1)} - \frac{R_{2m}\eta_m^L + \frac{1}{2}\xi_2^R}{\cosh^2(x_3 - x_1)}\,, \tag{39}$$

$$A_3 = \frac{1}{2}\,\frac{\xi_3^R}{\sinh^2(x_1 - x_2)} - \frac{R_{3m}\eta_m^L + \frac{1}{2}\xi_3^R}{\cosh^2(x_1 - x_2)}\,. \tag{40}$$

4. CONCLUDING REMARKS

In this talk, we have discussed the generalization of the Euler-Calogero-Sutherland model by introducing two internal variables "spin" and "isospin", using the integrable model based on the general matrix group $GL(n, \mathbb{R})$. We outline its relation to the well-known integrable model. Our consideration confirms once more that the clue to an integrability of a model is often hidden in the possibility to connect it with a known higher-dimensional exactly solvable system by its symplectic reduction to its invariant submanifold [4, 5]. A rich spectrum of these types of finite-dimensional models, obtained by the generalized "momentum map" is well-known (see e.g. [9]). Over the last decade it has been recognized that the same happens in the infinite-dimensional case. Integrable two-dimensional field theories have been found from the so-called WZNW theory applying the Hamiltonian reduction method [10]. An important class of finite-dimensional systems was discovered by the Hamiltonian reduction method from the so-called matrix models (for a recent review see e.g.[11]). The interest to this type of models has a long history starting with the Wigner study of the statistical theory of energy levels of complex nuclear system [12]. Nowadays we have revival of the interest to a matrix models connected with the search of relations between the supersymmetric Yang-Mills theory and integrable systems (for a modern review see e.g. [13]). The relation between the Euler-Calogero-Moser model and the $SU(2)$ Yang-Mills theory in long-wavelength approximation was obtained in ([14]).

Acknowledgments

It is a pleasure to thank the local organizers of the XXIII International Colloquium on Group Theoretical Methods in Physics and B. Dimitrov, V.I. Inozemtsev, A. Kvinikhidze, M.D. Mateev, and P. Sorba for illuminating discussions.

[1] F. Calogero, J. Math. Phys. **10** 2191 (1969); J. Math. Phys. **10** 2197 (1969); J. Math. Phys. **12** 419 (1972); B. Sutherland, Phys. Rev. **A 4** 2019 (1971); Phys. Rev. **A 5** 1372 (1972); J. Moser, Adv. Math. **16** 197 (1975).

[2] J. Gibbons and T. Hermsen, Physica **D 11** 337 (1984).

[3] S. Wojciechowski, Phys. Lett. **A 111** 101 (1985).

[4] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, (Springer-Verlag, Berlin-Heidelberg-New-York, 1978).

[5] J. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry.* (Springer-Verlag, Berlin-Heidelberg-New-York, 1994).

[6] D.R. Zelobenko, *Compact Lie groups and their representations* (Translations of Mathematical Monographs, v.40 AMS,1978).

[7] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University Press, New York, 1964).

[8] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, (Princeton University Press, Princeton, NJ, 1992).

[9] A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras* (Birkhäuser, Basel-Boston-Berlin, 1990).

[10] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** 1 (1992).

[11] A.P. Polychronakos, *Generalized statistics in one dimension* (hep-th/9902157).

[12] M.L Mehta, *Random matrices* (Academic Press, Boston,1991).

[13] E. D'Hoker and D.H. Phong, Lectures on supersymmetric Yang-Mills theory and integrable systems (hep-th/9912271).

[14] A.M. Khvedelidze and D.M. Mladenov, Phys. Rev. **D62** 125016 (2000); hep-th/990633.